Eur. Phys. J. Special Topics **223**, 2065–2085 (2014)

DOI: 10.1140/epjst/e2014-02250-7

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Human population and atmospheric carbon dioxide growth dynamics: Diagnostics for the future

A.D. Hüsler[a] and D. Sornette[b]

Department of Management, Technology and Economics, ETH Zurich, Scheuchzerstrasse 7, 8092 Zurich, Switzerland and Swiss Finance Institute c/o University of Geneva, 40 Blvd. du Pont d'Arve, 1211 Geneva 4, Switzerland

Received 22 April 2014 / Received in final form 18 August 2014

Published online 24 October 2014

Abstract. We analyze the growth rates of human population and of atmospheric carbon dioxide by comparing the relative merits of two benchmark models, the exponential law and the finite-time-singular (FTS) power law. The later results from positive feedbacks, either direct or mediated by other dynamical variables, as shown in our presentation of a simple endogenous macroeconomic dynamical growth model describing the growth dynamics of coupled processes involving human population (labor in economic terms), capital and technology (proxies by CO_2 emissions). Human population in the context of our energy intensive economies constitutes arguably the most important underlying driving variable of the content of carbon dioxide in the atmosphere. Using some of the best databases available, we perform empirical analyses confirming that the human population on Earth has been growing super-exponentially until the mid-1960s, followed by a decelerated sub-exponential growth, with a tendency to plateau at just an exponential growth in the last decade with an average growth rate of 1.0% per year. In contrast, we find that the content of carbon dioxide in the atmosphere has continued to accelerate super-exponentially until 1990, with a transition to a progressive deceleration since then, with an average growth rate of approximately 2% per year in the last decade. To go back to CO_2 atmosphere contents equal to or smaller than the level of 1990 as has been the broadly advertised goals of international treaties since 1990 requires herculean changes: from a dynamical point of view, the approximately exponential growth must not only turn to negative acceleration but also negative velocity to reverse the trend.

[a] e-mail: `andreas.huesler@live.com`
[b] e-mail: `dsornette@ethz.ch`

Today, humanity uses the equivalent of 1.5 planets to provide the resources we use and absorb our waste. This means it now takes the Earth one year and six months to regenerate what we use in a year.[1] *– Is humanity inevitably doomed?*

1 Introduction

During the 1960s, scientists and policy makers were most concerned about human population growth (see for instance [35]) and about depletion of energy resources (see for example the first report by the Club of Rome ([19]) and its recent reassessment by [13]). The growth rate[2] of human population peaked in the late 1960s and although population is still growing, it is unfortunately no longer the prime concern of policy leaders, who face a host of daunting more immediate issues since the financial crises, economic recessions, Arab spring and color revolutions.

More recently, scientists and politicians became aware of global warming (see [37] for a historic overview) due to or augmented by anthropogenic effects [25]. We focus here on the fact that due to the massive use of fossil energies the world economy emits, among many other products, large amounts of carbon dioxide go into the atmosphere. Part of this carbon dioxide is later absorbed by the oceans and plants. The fraction of carbon dioxide found in the atmosphere is currently around 50% of the total anthropogenic emissions, with a slight upward trend ([22]). Once in the atmosphere, this CO_2 is thought to play a pivotal role in global warming. In a recent Nature issue, Ref. [23] identifies climate change due to CO_2 emissions as one of the most pressing problems that mankind needs to address.

Ref. [36] discusses the IPAT identity, which partitions factors that are believed to drive carbon dioxide emissions. They contribute carbon dioxide emissions to three factors

$$I = P \cdot A' \cdot T, \tag{1}$$

where I (impact) denotes the carbon dioxide emissions, P is human population, A' represents the affluence (measured as gross world product per capita) and T is technology. The IPAT identity helps thinking about the contributions of different variables and has been extensively used and discussed in the literature (see for instance [5]) and will be used as a starting point. Equation (1) wants to mean that CO_2 emission (I) is a function of all three others factors ($PA'T$), which is much more general that the naive product expression of P, A' and T. More important, because one deals fundamentally with a complex dynamical system driven by entangled feedback loops with delays, the IPAT identity falls short, in our opinion, of providing the framework to understand the inter-relationships among the dynamical variables. It is especially important to develop an explicit dynamical framework with delays, when studying the time-evolution of global variables such as atmospheric carbon dioxide content and human population, as illustrated by Refs. [38–41].

Motivated by a dynamical view of the human-Earth system, we present here a framework borrowing from the theory of endogenous macroeconomic growth

[1] `http://www.footprintnetwork.org/en/index.php/GFN/page/world_footprint/`

[2] The growth rate r of the human population (or of any other variable) is defined by expression (2). Thus, a constant growth rate corresponds to a population growing exponentially, with a doubling time given by $(\log 2)/r$. As the present growth rate is $r(2011) \approx 1.0\%$ per year, this gives a present doubling time of 69 years. If nothing changes, the present 7 billion people (passed in the last quarter of 2011) will be more than 14 billion in 2080. This is in contradiction with projections of OECD for instance and other international organizations, which optimistically expect human population to stabilize around 9 to 10 billion individuals.

([18,24]), whose feedback loops are shown to generate robust regimes of super-exponential growth. Mathematically, these regimes can be described by simple equations, whose solutions exhibit finite-time singular (FTS) power law behaviors. The interest in such solutions is that they point to changes of regime (see also [10,15,26]).

Accelerating atmospheric carbon dioxide growth due to industrial activity has been previously reported in Ref. [4]. In contrast to Ref. [4], the present paper focuses on the interplay between positive feedbacks on carbon-dioxide growth due to population and technology in a global macroeconomic model. Reference [9] develops global circulation model based on thermal conservation equations. It is interesting to note that, in such a model, energy consumption is super-exponential as well.

The article is organized as follows. Section 2 contains a simple mathematical framework to model growth, first for a single variable like population in the presence of positive feedback, and then with several coupled variables, such as population, capital and technology. Two benchmark models, the exponential law and the FTS power law, are obtained as limiting cases of the theoretical framework. Section 3 presents the results of the calibration of the exponential law and the FTS power law models to some of the most extensive data sources on human population and atmospheric CO_2 content in the last two centuries up to present (our data sources are given in Appendix A). Section 4 summarises our main findings and concludes on the likely future outcomes.

2 Models

2.1 Exponential growth and generalization to super-exponential growth

The benchmark for population growth is the Malthus model, which postulates that population growth is proportional to the population itself, capturing the simple idea that the number of children is proportional to the number of parents:

$$\frac{dp}{dt} = r \cdot p(t). \tag{2}$$

The solution of equation (2) is the exponential function

$$p(t) = a' \exp(r \cdot t) + c', \tag{3}$$

where c' is usually set to zero for population analysis.

Historically, Eq. (2) has been improved by Verhulst (see [33] and [34]) into the logistic equation to account for the competition for scarce resources between individuals. This competition can be embodied into the quadratic term $-r[p(t)]^2/K$, where K is the carrying capacity, leading to the Verhulst or logistic equation,

$$\frac{dp}{dt} = r \cdot p(t) \left(1 - \frac{p(t)}{K}\right) \cdot \tag{4}$$

This negative feedback of the population on the growth rate $r \rightarrow r(1 - p(t)/K)$ leads to a cross over from the exponential growth for $p(t) \ll K$ to a saturation of the population at long times, which asymptotically converges to K. The full explicit solution of Eq. (4) is

$$p(t) = \frac{K p_0 \, e^{rt}}{K + p_0 \, (e^{rt} - 1)}, \tag{5}$$

where $p(0) := p_0$. This expression (5) encapsulates the initial exponential growth with rate r at early times when $p(t)$ is still much smaller than the carrying capacity K to an exponential convergence to K at long times. The typical time scale for convergence to K is also controlled in the logistic equation by the growth rate r. This results in the well-known S-shape growth with saturation, used to describe animal population dynamics as well as the penetration of commercial products, for instance. Verhulst thought that Malthus was wrong (and therefore over-pessimistic when comparing a supposed exponential human growth with limited food resources) not to take into account the negative feedbacks embodied in the quadratic term $-r[p(t)]^2/K$, that would lead naturally to an equilibrium.

But, the human population at the time of Verhulst and until around 1960 followed neither his specification, nor the Malthusian exponential growth. As reviewed in Refs. [1,15,16,35] and references therein, the human population grew faster than exponential, with the growth rate r itself growing.

The simplest generalization of equation (2) that accounts for this observation assumes that the growth rate r becomes $r \cdot [p(t)/p_0]^\delta$, where $\delta > 0$ and p_0 is some reference population. The positivity of δ captures the positive feedback of population on the growth rate: the larger the population, the larger the growth rate due to a larger pace of innovations in the agriculture and health sectors that promote growth. Equation (2) then transforms into

$$\frac{dp}{dt} = R \cdot p(t)^{1+\delta}, \tag{6}$$

where $R = r/p_0^\delta$. The solution of equation (6) reads

$$p(t) = \frac{p_0}{(1-(t/t_c))^{1/\delta}} \quad \text{if } \delta > 0, \quad \text{with } t_c = \frac{1}{R}\,\frac{1}{\delta}\,\frac{1}{p_0^\delta}, \tag{7}$$

As can be seen, the critical time t_c at which the solution diverges is determined from the parameters of equation (6) and the initial population p_0 at time $t = 0$. For $\delta \to 0$, we recover the exponential solution (3), since

$$p(t) = p_0 \ \exp\left[-\frac{1}{\delta}\ln(1-(t/t_c))\right] \approx p_0 \ \exp\left[\frac{t}{\delta t_c}\right] \to p_0 \ e^{Rt}, \ \text{for } \delta \to 0. \tag{8}$$

The standard exponential growth can thus be seen as the limit of a finite-time-singularity (FTS) power law with positive feedback exponent δ tending to zero. The singular solution (7) was first discussed by [35] (see [32] for assessments of the relative merits of the "natural science" versus the "demographic" approach, [18] for an economic underpinning that we explore later, and [15,17] for extensive generalizations). In ecology, the positive correlation between population density and the per capita population growth rate at the origin of the FTS behavior (7) is known as the Allee effect ([31]). More generally, Allee discovered the existence of an often present positive relationship between some component of individual fitness and either numbers or density of conspecifics. The Allee effect is usually used to refer to the self-reinforcing feedbacks that promote accelerated extinction of species and that can be modeled by finite-time crossing of zero (see [38,39] for detailed mathematical formulations). Finally, Ref. [12] provides a rigorous mathematical framework of ordinary differential equations with finite-time singularities, using a generalized version of equation (6), where the right hand side is replaced by an arbitrary polynomial of $p(t)$.

The use of the mathematics of FTS to describe and diagnose changes of regime is not new. For instance, we refer to Refs. [15,29] for population dynamics and financial markets, Ref. [26] for applications to engineering failures and earthquakes, Refs. [28,30] for a large variety of systems, Ref. [7] for climate systems, and Refs. [3,8,27] for environmental systems. These authors applied the concept of dynamical phase transitions and FTS behavior to different systems exhibiting a bifurcation, crisis, catastrophe or tipping point by showing how specific signatures can be used for advance warnings.

One can generalize (6) to take into account positive feedbacks of the growth rate $d \ln p/dt$ on its rate of change $d^2 \ln p/dt^2$ (see [14]), to arrive at solutions that exhibit FTS not in the variable $p(t)$, but in its derivative dp/dt. We will thus use the slightly more general expression encompassing these cases:

$$p_{\text{power}}(t) = a(t_c - t)^{-1/\delta} + c. \tag{9}$$

A FTS in dp/dt and not in $p(t)$ corresponds to $-\infty < \delta < -1$ such that $0 < -1/\delta < 1$, together with $a < 0$ for an increase up to the value $p_{\text{power}}(t_c) = c$. Here, the meaning of the exponent δ is different from its use in Eq. (6).

We shall use the *exponential model* (3) and the *power law model* (9), as our two competing hypotheses. The essential difference between the *exponential model* and the power law model is that the former is defined for all times, while the later is valid only up to a finite time, the critical time t_c beyond which the solution ceases to exist. The singular behavior at t_c is not meant to predict a genuine divergence but only, as already stressed, that the system is exhibiting a transition to a qualitatively new regime. The mathematical divergence does not occur because the system is then subjected to additional mechanisms neglected until they become dominant as the variable grows in an accelerated manner.

2.2 Properties distinguishing the exponential and the hyperbolic power law model

Heated discussions among demographers greeted the publication of von Foerster et al. [35] concerning the singular solution (7): the demographers criticized the use of mathematical models such as (6) as perhaps the clearest illustration of how bad use of mathematics may yield senseless results; actually, what the demographers missed was that the FTS should not be taken at face value, but as the signature of a transition to a new regime. Singularities do not exist in natural and social systems, but the singularities of our mathematical models, which are approximate representations of reality, are usually very good diagnostics of the changes of regime that occur in these systems ([28]). The perhaps clearest examples are the phase transitions between different states of matter (solid-liquid-gas-plasma, magnetized to non-magnetized, and so on) that statistical physics describes so well with its classification involving the nature of the singularity exhibited by the free energy of the system ([11]).

As t approaches t_c from below, two regimes can be observed for the power law model:

$\delta < 0$: $(t_c - t)^{-1/\delta}$ goes to zero for $t \to t_c$ and $p_{\text{power}} \to c$.
$\delta > 0$: $(t_c - t)^{-1/\delta}$ goes to infinity for $t \to t_c$ and $p_{\text{power}} \to \text{sign}(a) \cdot \infty$.

Figure 1 illustrates the qualitatively different behaviors allowing one to distinguish between the linear growth model ($dp(t)/dt \sim t$), the *exponential model* (3) and the *power law model* (9), in different standard plot representations. We present the same functions in four different representations in order to help develop an intuition based on the qualitative shape of data (linear, convex versus concave) to diagnose non-parametrically the qualitative nature of the growth. For instance, plot (b) (linear in the abscissa and logarithmic in the ordinate) is very useful to immediately get the

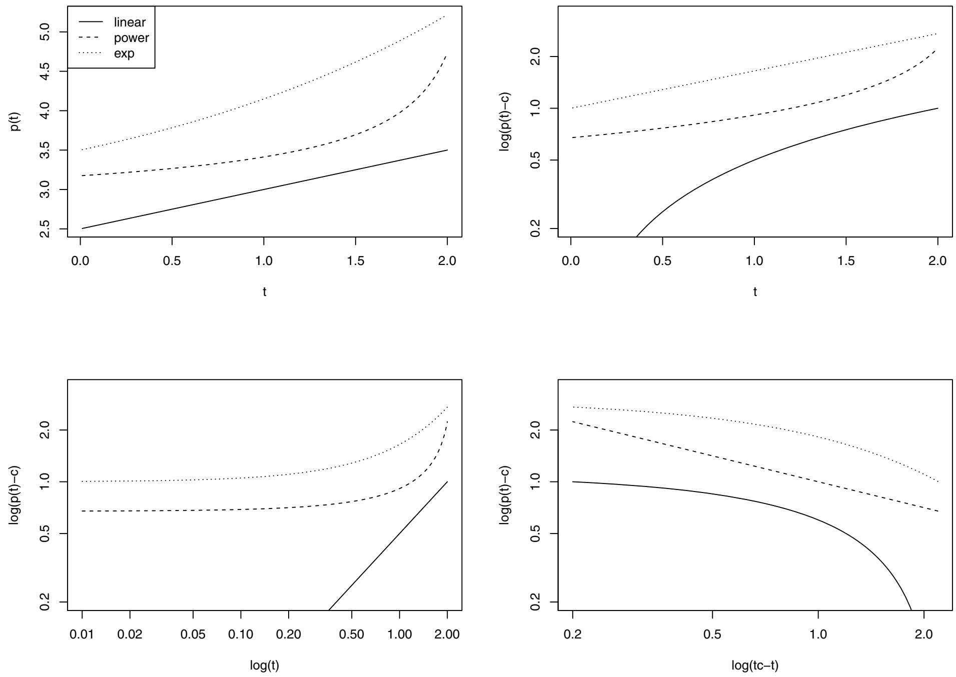

Fig. 1. Illustration of the qualitatively different behaviors of the *exponential model* (3), the *power law model* (7) and a *linear model*, in different standard plot representations. For each of the four plots, the linear function $0.5t + 3.25$ is compared with the exponential function $e^{0.5t} + 2.5$ and with the hyperbolic power law $(2.2 - t)^{-0.5} + 2.5$. (a) Is linear-linear, (b) is linear-log, (c) is log-log and (d) is log-log referenced to the singularity. The constant c is set to 2.5. The relative vertical positions of the three curves are arbitrarily chosen (from the above values) for the sake of a clear visualization (see Appendix for further discussion).

intuition on whether the growth is approximately exponential (linear in this representation), slower (concave) or faster (convex) than exponential. Similarly, the other plot representations allow one to immediately determine the qualitative nature of the growth and confirm the interpretation obtained from the other plot representations.

Depending on the scale of the abscissa and the ordinate, exponential growth and finite-time singular (FTS) power-law growth can look very different, as illustrated in Fig. 1:

(a) the linear-linear plot shows the dual property of the FTS power law function, which is to both grow initially slower than the two other models, and then to catch up explosively.

(b) In the linear-log plot, by construction, the exponential function is a straight-line, thus a linear dependence in this representation qualifies an exponential growth. The linear model is concave (slower growth) and the power law FTS model is convex (faster growth).

(c) The log-log plot would qualify a power law t^{β} as a straight line whose slope is the exponent β. Hence the linear function is also linear in this representation with slope 1. Both the exponential and FTS power law model exhibit an upward convex shape. It is important not to confuse a power law and a FTS power law (or hyperbolic power law): the former is proportional to a power of t and thus exists for all times, while the later is proportional to a power of $t_c - t$ and is only defined for $t < t_c$.

(d) In the log-log plot in the variable $t_c - t$, by construction, the FTS power law is qualified by a straight line behavior, with a slope equal to the exponent $-1/\delta$. Both linear and exponential models are associated with concave curves, characterizing a slower growth in the vicinity of t_c. Note that time t increases from right to left since the abscissa plots the logarithm of $t_c - t$.

2.3 Faster-than-exponential growth by feedbacks between macro-economic variables

Up to now, we have postulated the form (6) to capture the possible existence of a positive feedback of population on the population growth rate. Such a simplified ansatz leaves two issues unresolved. First, the positive feedback of population on growth rate may not be direct, but mediated by other variables via indirect mechanisms. Second, the consequences on the dynamics of carbon dioxide emissions are not clear. We thus address these two issues using an economic framework developed by Kremer [18], following the approach of Ref. [15]. The following derivation is not intended to represent a faithful economic growth model that we would like to promote, but is offered to illustrate the importance of indirect mechanisms in growth processes. In particular, we would like to stress the fact that faster-than-exponential growth is a robust outcome of multi-dimensional loop processes. Even when each feedback process individually leads to an exponential or even a subdued sub-exponential growth, the overall dynamics can be super-exponential.

In economics, population $p(t)$ translates into labor force $L(t)$, which is assumed to be proportional to population. In addition to population represented by the labor force, we consider the effect of technology level $A(t)$ and of the amount $K(t)$ of available capital. In the presence of labor and capital, with a given technology level, the economy is going to produce an output $Y(t)$, for instance proxied by GDP. In the macroeconomics of endogenous growth ([24]), it is common to use the Cobb-Douglas equation (originally developed in [6] and extensively discussed in [24]) to relate the total output to labor, capital and technology as follows[3]:

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}, \text{with} \quad 0 < \alpha < 1. \tag{10}$$

Furthermore, we use the assumption by Solow that a constant fraction $0 < s < 1$ of the economy goes to savings, i.e. capital grows according to

$$\frac{dK}{dt} = sY(t). \tag{11}$$

Following [18], we assume that, as already mentioned, labor is proportional to capital

$$K(t) \sim L(t). \tag{12}$$

We further assume that technology change is depending on capital, labor and current level of technology according to

$$\frac{dA}{dt} = d\ K(t)^{\eta} \times L(t)^{\gamma} \times A(t)^{\theta}, \tag{13}$$

[3] A' in the IPAT equations stands for gross world product per capita, whereas in the Cobb-Douglas equation A stands for technology. Further, the IPAT equation uses T instead of A to denote technology. Similar, the macro-economists refer to L as labor, whereas P in the IPAT equality stands for population. We will not distinguish between labor L and population P and use the terms interchangeably.

where the exponents η, γ and θ are all positive, expressing a positive feedback effect of each of the variables on the growth of technology and d is a positive constant. Putting together all these ingredients, we can rewrite the Kremer (12) and Solow (11) equations as a system of two coupled ordinary differential equations:

$$\frac{dA}{dt} = eL(t)^{\eta+\gamma} \times A(t)^{\theta}, \tag{14}$$

$$\frac{dL}{dt} = fL(t) \times A(t)^{1-\alpha}, \tag{15}$$

where e and f are two positive constants. Equation (15) basically states that labor (and thus population) is growing exponentially, holding technology constant. In other words, the growth rate of population is controlled by a nonlinear function of technology. Here, this nonlinear function is a power law with exponent $0 < 1 - \alpha < 1$, which embodies the benefits that technology brings in decreasing death rates, for instance via improvement in health care, or in feeding more mouths. Invoking these mechanisms is standard in demographic research.

We look for solutions exhibiting a FTS of the form

$$A(t) = A_0(t_c - t)^{-1/\mu}, \tag{16}$$

$$L(t) = L_0(t_c - t)^{-1/\kappa}. \tag{17}$$

Note that the critical time t_c of the singularity, if it exists, is necessarily the same for both variables, as seen from inspection of the two coupled Eqs. ((14),(15)). Inserting this ansatz in Eqs. ((14),(15)), we obtain a system of linear equations for the unknown inverse exponents $1/\mu$ and $1/\kappa$, whose solutions read

$$\mu = 1 - \alpha, \tag{18}$$

$$\kappa = \frac{\eta + \gamma}{2 - \theta - \alpha}(1 - \alpha). \tag{19}$$

The condition for the solutions (16,17) to hold is that μ and κ be strictly positive. This implies $0 < \alpha < 1$ and $\alpha < 2-\theta$. If $\theta \leqslant 1$, then the conditions are always satisfied in the regime where the Cobb-Douglas equation holds. The case $\theta \leqslant 1$ is particularly interesting because it corresponds to a sub-exponential growth of technology, for a fixed labor force. In other words, for a fixed population level, Eq. (14) gives a long-time growth of the form $A(t) \sim t^{\frac{1}{1-\theta}}$, which is sub-exponential (slower than exponential) for $\theta < 1$ and exactly exponential for $\theta = 1$. It is the coupling between a sub-exponential growth of $A(t)$ and an exponential growth of population $L(t)$ mediated by nonlinear feedback loops that create the super-exponential finite-time singularity. This behavior underlines the possible traps of single variable analysis.

These results can be translated into a prediction of carbon dioxide emission via the following simple assumption. Assuming that carbon dioxide emissions are proportional to production divided by some power of technology ξ, we have

$$\frac{dC}{dt} = a\frac{Y(t)}{A(t)^{\xi}} = h(t_c - t)^{-1/\varphi}, \tag{20}$$

where $\varphi = (1/\kappa - \xi/\mu + 1)^{-1}$ (see appendix for details of the derivation) and C stands for the total carbon dioxide content in the atmosphere. The constant a absorbs the dimensional relations between the different variables. The introduction of a non-zero exponent ξ accounts for the common observation that more developed countries tend to have a lower footprint and smaller carbon emissions per unit of output, due to the progressive adoption of more efficient technologies and the increasing importance of a clean environment in the utility functions of consumers.

Let us thus stress the main result of this exercise. We have $\frac{dA}{dt} \sim A(t)^{\theta}$ at fixed labor with $\theta < 1$ and $\frac{dL}{dt} \sim L(t)$ at fixed technology. Thus, there is no way to get a faster-than-exponential growth in any of these two variables alone. However, when coupling them via the feedback of labor on technology and that of technology on labor, the FTS power law solutions ((16),(17)) emerge. Hence, a finite-time singularity can be created from the interplay of several growing variables resulting in a non-trivial behavior: the interplay between different quantities may produce an "explosion" in the population even though the individual dynamics do not!

Of course, infinities do not exist on a finite Earth! These singularities should not be interpreted as the prediction of real "blow-ups". They can be however faithful description of the transient dynamics up to a neighborhood of the predicted critical time t_c. Around t_c, new mechanisms kick in and produce a *change of regime*.

To illustrate the above point, let us go through a detailed scenario where the individual processes stay finite in finite time, but the combination via feedback can lead to finite time singularities. Consider the following parameters

$\alpha = \frac{1}{4}$: As in the seminal paper [6].

$\theta = 1$: Linear feedback from technology A on itself. Holding all other factors constant, technology will grow exponentially (see Eq. (13)).

$\eta + \gamma = 1$: The simplest possible, non-trivial, assumption.

With these numbers, we obtain the two exponents $\mu = 3/4$ and $\kappa = 1$ for the Eqs. (16) and (17), respectively, and the value $1/\varphi = 5/3$ for the rate of carbon dioxide emission given by Eq. (20), assuming carbon dioxide emission per capita technology is as efficient as general technology A, i.e. $\alpha = \xi = 1/4$. Although, we have only assumed exponential growth of all individual factors, carbon dioxide emission is predicted in this example to grow faster than exponential, leading to a mathematical FTS which is the signature of a non-sustainable regime towards a new behavior (see Fig. 2). We refer to [38–41] for detailed classifications of possible regimes.

Even less stringent conditions for a FTS to occur are needed when the description of the dynamics of the system in terms of two coupled Eqs. ((14),(15)) is augmented to take into account the dynamics of additional coupled variables, leading to systems of three or four coupled equations. Such additional positive feedback loops include non-linear lagged dependencies of capital on labor (thus extending Kremer's simplifying assumption (12)).

3 Empirical data analysis of human population and CO_2 atmospheric content

3.1 Dynamics of human population

Population growth at preindustrial levels is generally estimated to be below 0.5%. Then, starting with the industrial revolution, the human population growth rate grew rapidly as can be observed in Fig. 3, until the late 1950s. A sharp decrease of the growth rate then occurred, then followed by resumed acceleration until its peak in 1964 at about 2.4% per year, from which a slow decrease of the growth rate can be observed.

The first regime until about the mid-1960 is incompatible with the exponential model, which corresponds to a constant growth rate. Over the time period 1850 to

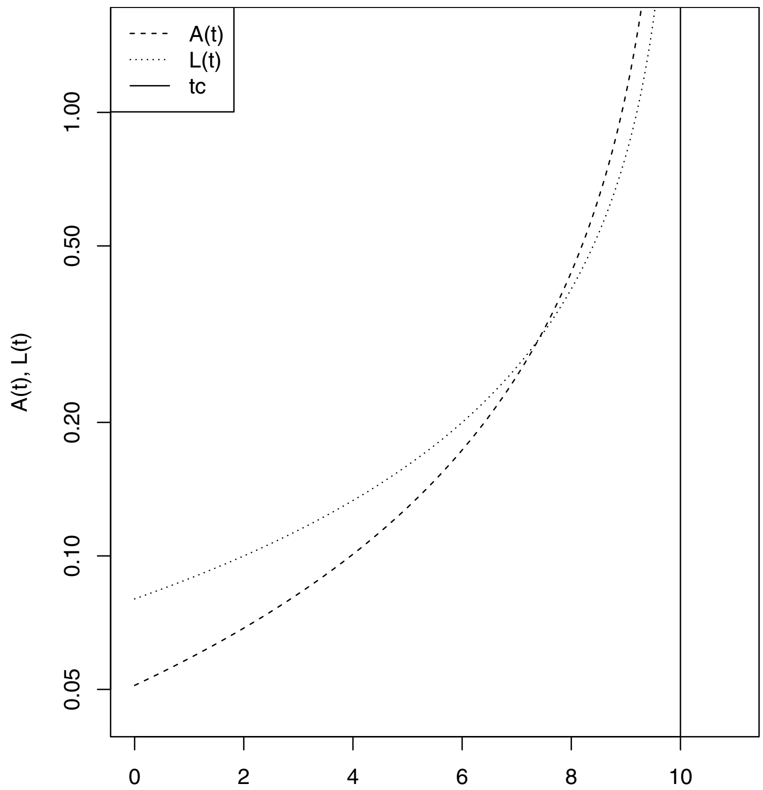

Fig. 2. Numerical solution of Eqs. (14) and (15) with $\alpha = \frac{1}{4}$, $\eta + \gamma = 1$, $\theta = 1$ and $t_c = 10$. The initial conditions are $A(0) = 1.1$ and $L(0) = 0.8$. We assume without loss of generality $e = f = 1$, as these coefficients can be absorbed in the units of A and L respectively. $L(t)$ and $A(t)$ grow super-exponentially towards a singularity occurring at the same time as a result of their coupling. $A(t)$ and $L(t)$ are plotted on a semi-log plot as function of time. The upward curvatures and approaches to the singular vertical asymptote exemplify the super-exponential growth.

1965, the exponential model is greatly inferior to the FTS power law model (the ratio of squared errors between the power-law and the exponential fits is 0.017). Using model (9), we estimate that the growth exponent δ is approximately equal to 2, that is, even larger than the value 1 estimated in [35]: clearly, population growth over this time period was faster than exponential and the FTS power law model accounts parsimoniously for the data. Specifically, the fitted parameters are $\delta = 2.0$ and $t_c = 1988$ CE for the power-law and $r = 1.1\%$ per year for the exponential fit.

This growth of the population growth rate is a straightforward signature of a super-exponential growth of the population, i.e., faster than exponential, as visualised in Fig. 4. Recall that the logarithmic scale of the population as a function of linear time qualifies an exponential growth (i.e. with a constant growth rate) as being a straight line whose slope is the growth rate. In contrast, one can observe a clear upward curvature, again exemplifying the fact that the growth rate has been growing until the 1960s. One billion people in Earth was passed in around 1804, three billions in the mid-1950s and seven billions in 2011.

After 1965, the population growth rate stopped growing and in fact started falling gradually. This deceleration can be seen in Fig. 3 as well as in Fig. 5. The slight concave shape of the data in the semi-logarithmic plot is another visual evidence of the slow

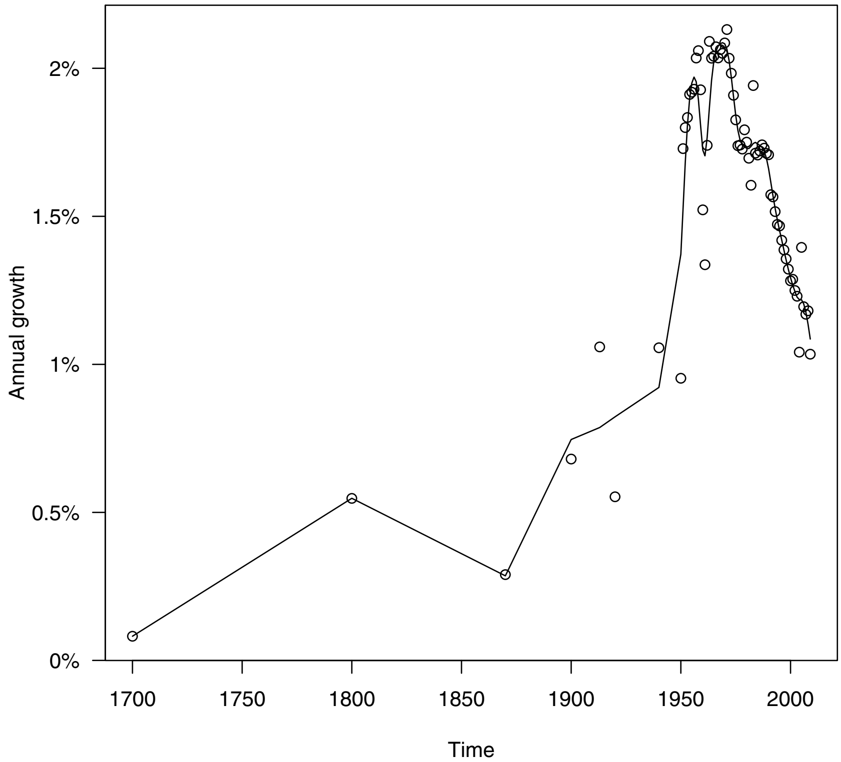

Fig. 3. Annualized world population growth rate from year 1800 to 2010, providing a direct diagnostic of a super-exponential growth until the 1960s characterised, as explained in the text, by a growth of the growth rate. The empty circles are the data points and the continuous line has been obtained by using a smoothing filter.

decrease of the population growth rate. However, over the whole time period from 1970 to 2010, performing a "horse race" between the exponential model (3) and the FTS power law model (9), we find that they are essentially indistinguishable, both visually and with respect to their summed square errors. The power law fit gives $\delta = 4.6$ and $t_c = 4312$ CE and the exponential fit gives $r = 1.5\%$ per year. This effective average growth rate from 1970 to 2010 is higher than the present estimated value around 1.0% per year. The US census bureau estimates that growth rates should continue to fall more or less linearly and fall below 0.5% per year at around 2050[4]. In contrast, our evaluations suggest, at least for a while, a plateauing of the growth rate at the present level of 1.0% per year as shown in Fig. 5 with two exponential fits, the first one from 1960 to 1990 with a constant growth rate of 1.7% per year, and the second one from 1990 to 2010 with a constant growth rate of 1.0% per year. At present, it is too early to determine whether the plateau of the growth rate r at 1% per year will continue or will transition to a resumed decrease. A disaggregated analysis is necessary at the level of countries, or even better regions, taking into account the large heterogeneities in the birth and death rates of the thousands of ethnies populating the planet.

Our focus in this section to the dynamics of the human population is motivated, as explained in Sect. 2, by the fact that, in the end, it is the total number of

[4] `http://www.census.gov/population/international/data/worldpop/table_population.php`

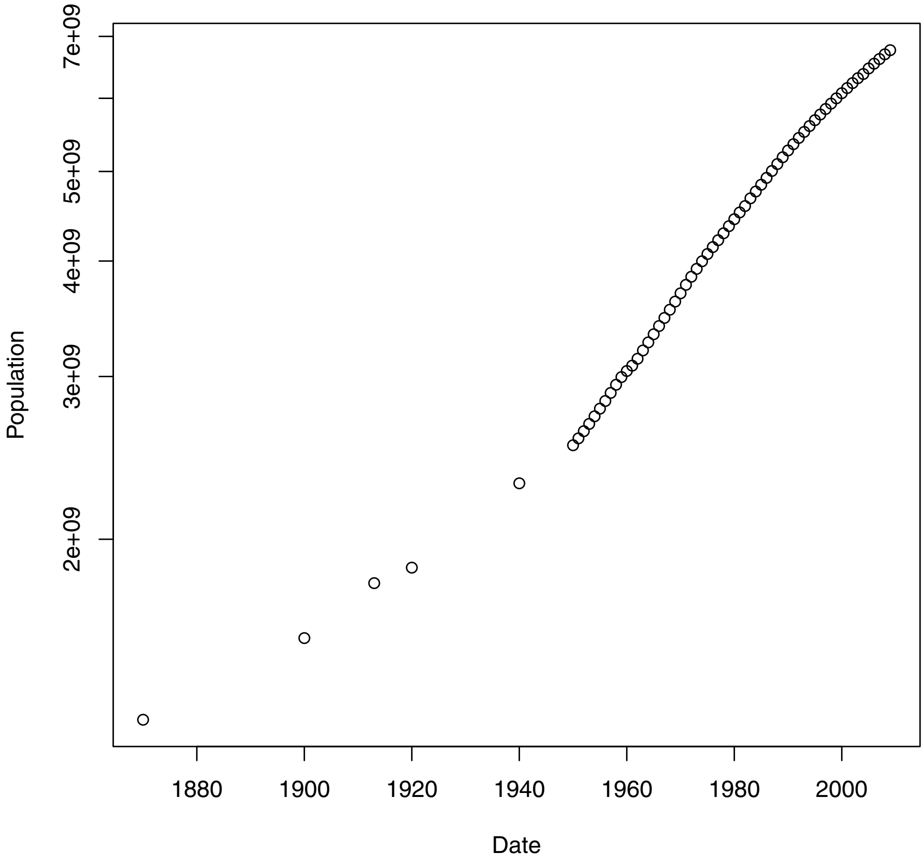

Fig. 4. Population (data compiled by Angus Maddison) since 1875.

people compounded by their individual environmental footprint that control the CO_2 atmospheric content in the atmosphere (as well as other environmental impacts and resource extractions). Our analyses suggest that the expected slowing down of the population to a maximum asymptotic value of 9 billions, as was extrapolated just one or two decades ago by various international organisations, has been really over-optimistic. At the present rate of 1.0% per year, if the rate does not abate, we would reach 10.5 billions in 2050 and more than 17 billions in 2100. While we do not believe that such a scenario is likely, we are on the trajectory to pass 8 billions around 2020, and probably 9 billions around 2030–2035, which are really just tomorrow!

3.2 Dynamics of CO_2 atmospheric content

In the recent human pre-industrial history (i.e. in the last thousand years), the level of CO_2 in the atmosphere has been approximately constant at a level of 280 ppm (parts per million). Starting with the industrial revolution in the 18th century, people have been using fossil fuels (first coal and later oil) on a large scale. Starting from about 1750, the dramatic acceleration trend of atmospheric CO_2 levels due to anthropogenic forcing is clearly observed (see Fig. 6): the flat pre-industrial level of around 280 ppm is replaced by higher levels reaching currently values just shy of 400 ppm.

We calibrate the *exponential model* (3) and the *power law model* (9) separately to two time periods: (i) from 1850 to 1954, for which the data originates from ice drill

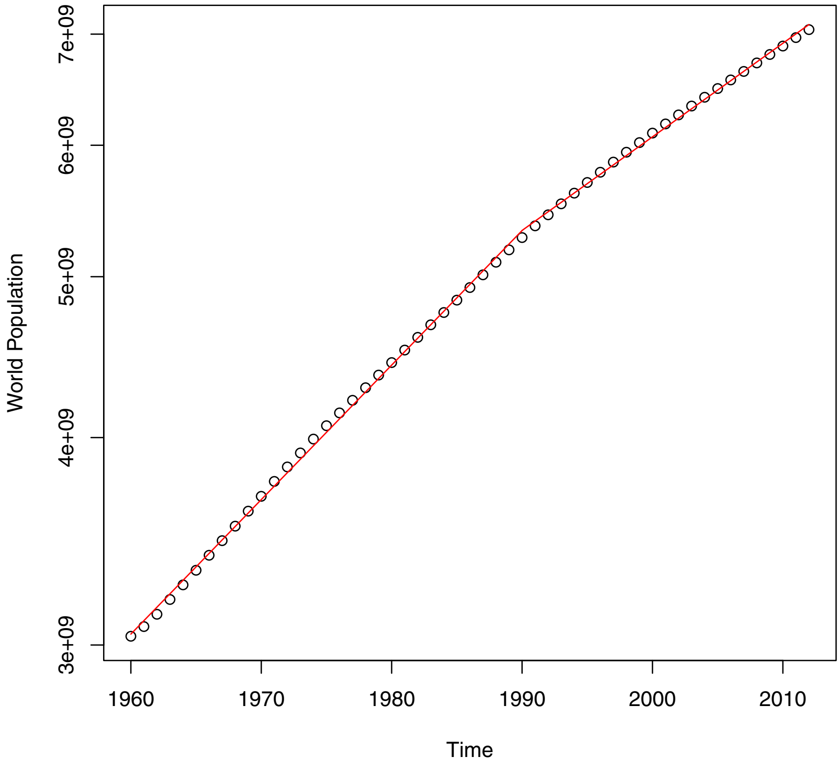

Fig. 5. Population estimated by the World Bank since 1960 on log-scale. The slight concave shape of the curve shows on the log plot shows that the growth rates are falling. Two exponential fits are shown, the first one from 1960 to 1990 with a constant growth rate of 1.7%, and the second one from 1990 to 2010 with a constant growth rate of 1.0%.

cores and (ii) from 1959 to 2011, for which the data originates from air samples[5]. The quality of the fits by the two models, as quantified by the sum of squared errors between theory and data, is practically equivalent. From 1959 to 2011, the fitted parameters are $\delta = 0.65$ and $t_c = 2304$ CE for the power-law and $r = 0.66\%$ per year for the exponential fit. (Corresponding to a doubling time of about 281 years according to the exponential fit.) From 1959 to 2011, the fitted parameters are $\delta = 0.71$ and $t_c = 2141$ CE for the power-law and $r = 1.6\%$ per year for the exponential fit (corresponding to a doubling time of about 85 years according to the exponential fit). Therefore, we cannot reject the hypothesis that the exponential model is sufficient to explain the data for each time window *separately.* However, the growth rate r calibrated with the exponential model (3) has more than doubled from the first period $1850 - 1954$ ($r = 0.66\%$ per year) to the second period of $1959 - 2011$ ($r = 1.6\%$ per year). While being not fully warranted given the heterogeneity of the data sources, we have also fitted the two models to the whole period from 1850 to 2011. We find that the FTS power law is the clear winner with a ratio of squared errors between the power-law and the exponential-fit equal to 0.92. The fitted parameters are $\delta = 0.25$ and $t_c = 2167$ CE for the power-law and $r = 2.4\%$ for the exponential fit (corresponding to a doubling time of about 64 years according to the exponential fit). Together with the more than doubling of the growth rate r from the first to the second

[5] We leave out the somehow turbulent years of 1941–1954, which are distorted due to WWII and its aftermath.

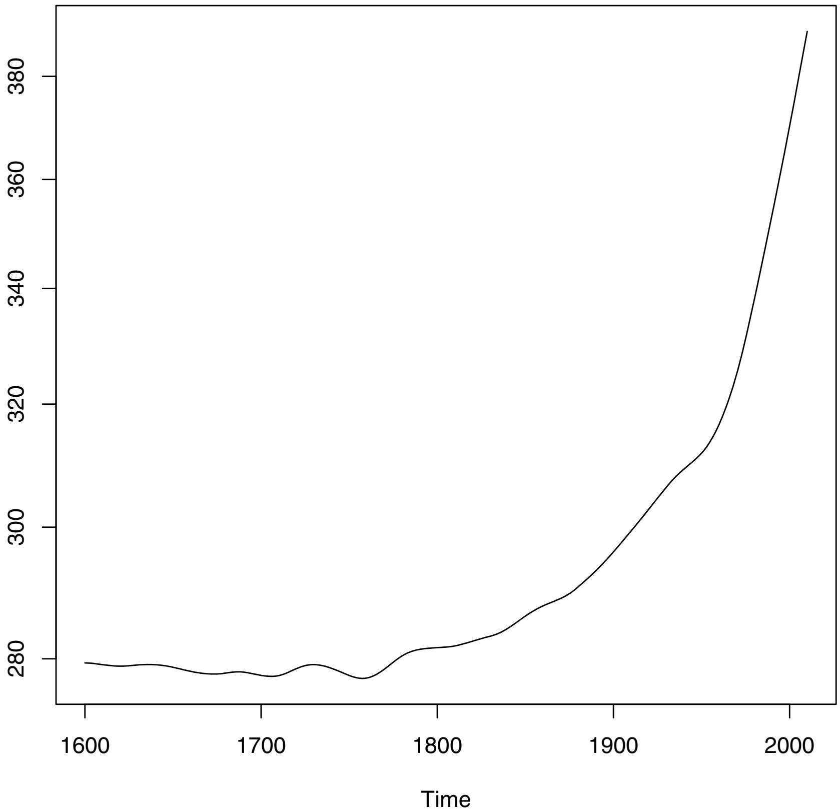

Fig. 6. Atmospheric CO_2 (parts per million) as a function of time since 1600 to 2011. The data shown combines ice core and air measurements from different sources. Before 1958, only incomplete data from ice-core is available. We have interpolated the missing data with a cubic spline. See data section for more details. A clear acceleration of the levels starting around 1800 can be observed.

time intervals, this result suggests the existence of a faster-than-exponential growth of the atmospheric content of carbon dioxide as a broad description of the period 1850 to 2011. Extrapolating this growth into the future, the CO_2 content would have exploded in finite time. However, in recent decades, a very progressive deceleration can be observed, as we now document.

For this, we attempt to be more precise on the nature and evolution of the CO_2 dynamics by estimating the growth rate b of equation

$$\ln(C - 280) = bt + c, \tag{21}$$

applied to the monthly data from the Mauna Loa site since 1958 when direct air measurements started, as it is considered to be one of the most reliable and longest time-series. Again, C represents the total atmospheric content of CO_2 in the atmosphere. We consider various time intervals $[t_1, t_2]$ in order to test for the robustness and trend of the estimated growth rate b's. Figure 7 shows a first coarse-grained analysis obtained by segmenting the period from 1958 to 2011 in two windows, one from 1958 to 1990 and the second one from 1995 to 2011. A fit of the exponential model (21) separately to each time window gives $b \approx 2.5\%$ per year until 1990 and $b \approx 2.0\%$ per year thereafter. This behaviour is refined and confirmed in table 8, which presents the estimated growth rates b of anthropogenic CO_2 defined in expression (21) in different

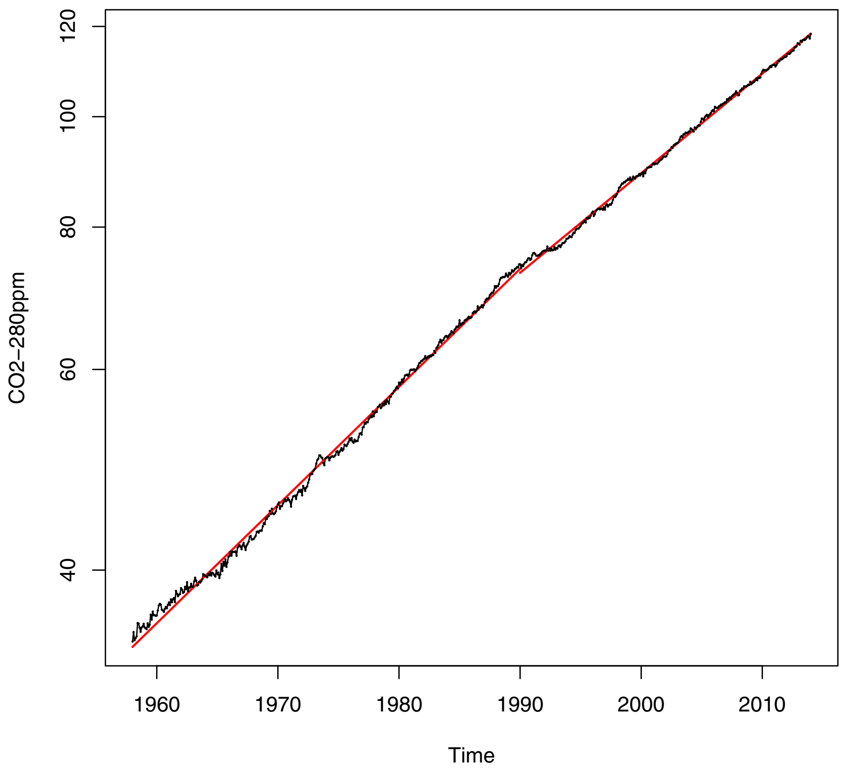

Fig. 7. Monthly CO_2 measurements at Mauna Loa since 1958 (when direct air measurements started). The y-axis is plotted on a logarithmic scale taking 280ppm as the baseline (i.e. the plot shows CO_2 minus 280 ppm in logarithmic scale). One can observe two regimes, here fitted by two different exponential growth rates (represented as two different straight lines in this linear-log plot), one from 1958 to 1990 and the second from 1995 to 2011. The estimates of the slope parameters for different time windows can be found in Fig. 8. They are approximately $\approx 2.5\%$ per year until 1990 and $\approx 2.0\%$ per year since.

	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010
1960	1.64%	2.09%	2.24%	2.33%	2.39%	2.41%	2.36%	2.31%	2.27%	2.24%
1965	NA	2.58%	2.46%	2.47%	2.50%	2.49%	2.39%	2.32%	2.27%	2.23%
1970	NA	NA	2.42%	2.49%	2.52%	2.50%	2.36%	2.28%	2.22%	2.19%
1975	NA	NA	NA	2.84%	2.62%	2.52%	2.31%	2.21%	2.17%	2.13%
1980	NA	NA	NA	NA	2.35%	2.40%	2.13%	2.08%	2.06%	2.05%
1985	NA	NA	NA	NA	NA	2.46%	1.92%	1.96%	2.00%	2.01%
1990	NA	NA	NA	NA	NA	NA	1.62%	1.96%	2.02%	2.03%
1995	NA	NA	NA	NA	NA	NA	NA	2.12%	2.09%	2.05%
2000	NA	NA	NA	NA	NA	NA	NA	NA	2.22%	2.05%
2005	NA	NA	NA	NA	NA	NA	NA	NA	NA	1.86%

Fig. 8. Estimation of the growth rates b of anthropogenic CO_2 defined in expression (21) in time windows $[t_1, t_2]$, where t_1 is given in the first left column and t_2 is indicated in the first top row. The heat map helps the eye identify a transition from an average growth rate $b \approx 2.5\%$ per year until about 1990 to a value $b \approx 2.0\%$ per year thereafter, confirming the evidence shown in Fig. 7.

time windows $[t_1, t_2]$, where t_1 is given in the first left column and t_2 is indicated in the first top row. The heat map helps the eye identify a broad transition from an average growth rate $b \approx 2.5\%$ per year until about 1990 to a value $b \approx 2.0\%$ per year thereafter. At a finer level, table 8 also shows that the super-exponential growth of the CO_2 content continued until about 1990, as the estimated growth rates b in the $[t_1, t_2]$ windows continued to grow to culminate at approximately 2.5% per year in $t_2 = 1990$ CE.

Studies of global warming are surrounded in controversy. Models are by force simplified imperfect representations of the systems they are supposed to capture. In this article, we have made the choice to use reduced form models involving just a few aggregate variables, in contrast to the very elaborated general circulation models used by the scientists contributing in particularly to the IPCC. Our choice can be criticised but has the advantage of being less prone to the instabilities well-known to plague large models with many adjustable parameters. Our analysis would benefit from the development of models of intermediate complexity, as done in climate science, connecting the low dimensional approach we have chosen with the very high dimensional parametric models used for instance by IPCC researchers.

4 Summary and concluding remarks

We have proposed a simple framework to think about growth dynamics of coupled processes involving human population (labor in economic terms), capital and technology (proxies by CO_2 emissions), based on endogenous economic growth theory. We showed that the positive feedback loops between several variables, such as population, technology and capital can give rise to the observed hyperbolic power law behavior ending with a finite-time singularity (FTS), notwithstanding the fact that the dynamics of each variable would be stable or at most exponential, conditional on the constancy of the other variables. It is the joint growth of the coupled variables and their mutual feedback loops that may give rise to the enormous acceleration characterized by the FTS behavior both in the equations and in the carbon dioxide content in the atmosphere, until recently.

Using some of the best databases available, we have performed empirical analyses showing that the human population on Earth has been growing super-exponentially until the mid-1960s, followed by a decelerated sub-exponential growth, with a tendency to plateau at just an exponential growth in the last decade with an average growth rate of 1.0% per year. The human population constitutes arguably the most important underlying driving variable of the content of carbon dioxide in the atmosphere. In contrast, we find that the content of carbon dioxide in the atmosphere has continued to accelerate super-exponentially until 1990, with a transition to a progressive deceleration since then, with an average growth rate of approximately 2% per year in the last decade.

Thus, until the 1960s, both population and atmospheric carbon dioxide content were super-exponentially accelerating in accordance with expressions ((16),(17)). Then, the slowing down from super-exponential to sub-exponential and then just exponential growth of the human population could be interpreted as a finite-size effect that has started to be felt for this variable earlier, as physical limits are more stringent for the human carrying capacity and the response of human birth and death rates to policies than they are for carbon dioxide emissions. The delayed slowdown of carbon dioxide atmospheric content, starting to slightly abate since 1990, suggests that the technological advances have not yet had a significant impact. This statement may appear shocking and counter-factual for developed countries. But, at the scale of the whole planet, one can observe that improvements in energy efficiency tending to reduce carbon emissions in the developed countries are counteracted by the increases of carbon emissions in some major developing countries [21], such as China and India, which use carbon emission inefficient technologies (for instance heavily based on coal burning). Indeed, consider the flagship publication in 2011 of the the International

Energy Agency (IEA) on the World energy outlook [20]. The IEA reports that carbon dioxide emissions jumped by 5.3% in 2010 to the record 30.4 gigatons, due mainly to increasing demand for coal in particular by China and India. The IEA raised its forecast for primary energy demand by a third between 2010 and 2035. The IEA report writes that, if there is no stringent new action by 2017, the energy related infrastructure in place would generate up to 2035 all the CO_2 emissions allowed over all times for the World to meet its target of a maximum temperature increase of 2C. And one should also note that even the USA and European countries are still using coal intensively as one of their energy sources.

Our results suggest to address next the question of estimating how much longer can the present situation (of exponential growth of the human population and of the atmospheric CO_2 content) last till the atmosphere is no longer suitable for human communities. The Intergovernmental Panel on Climate Change (IPCC) in its Fifth Assessment Report (AR5)[6] provides a vivid illustration of the complexity of this question, involving in particular the different time scales of adaptation of various ecosystems. Moreover, Canada and Siberia might welcome a significant global warming that might transform them in the new cereal providers of the world, while other world regions might go through disastrous droughts and/or floods. The most dangerous scenario is arguably a run-away positive feedback in which global warming unfreezes the long-trapped methane contained in the permafrost of the East Siberian Arctic Shelf, which might ultimately turn the planet Earth into a Venus like furnace.

In conclusion, the overall evidence presented in this paper suggests a cautious optimistic trend. We have reported first signs of improvements as the population and atmospheric CO_2 growth rates have slightly decelerated since their peaks respectively in the mid-1960s and 1990. But to go back to CO_2 atmosphere contents equal to or smaller than the level of 1990 as has been the broadly advertised goals of international treaties since 1990 requires herculean changes: from a dynamical point of view, the approximately exponential growth must not only decelerate but then turn to a negative velocity to reverse the trend. Given the continuing quasi-exponential growth of the human population and the rush for developing countries to catch up with level of consumption and therefore of energy use per inhabitant on par with developed countries, this seems to be an impossible task, short of an international consensus previously approached only at times of great global distress such as during World wars. Moreover, given the focus on rebooting economies after the great recession in 2009 and the focus on unemployment and economic competition, the centre of attention of the world is no more on these problems in the present decade.

We are grateful for stimulating discussion with Nicolas Gruber at an early stage of this work. We acknowledge financial support from the ETH Competence Center "Coping with Crises in Complex Socio-Economic Systems" (CCSS) through ETH Research Grant CH1-01-08-2 and ETH Zurich Foundation. We are thankful to Ryan Woodard for helping with the manuscript.

This article is contributed as an expression of our consideration and respect to our friend and colleague at ETH Zurich, Hans Jürgen Herrmann. In the second half of your life that has just started, please help us, Hans, address some of the issues discussed here with your great energy and skills!

A Data

Population data was obtained from Maddison Project website (`http://www.ggdc.net/maddison/maddison-project/home.htm`) and the World Bank

[6] `http://www.ipcc.ch/report/ar5/index.shtml`

(http://data.worldbank.org/indicator/SP.POP.TOTL) as well as from the website of the United Nations (http://www.un.org/esa/population/publications/sixbillion/sixbilpart1.pdf) and the website of U.S. Census Bureau (http://www.census.gov/population/international/data/idb/worldpoptotal.php). See also the overview at http://www.census.gov/ipc/www/worldhis.html.

Detailed data on the quantity of carbon dioxide in the atmosphere is available since 1959, when air measurements began at Mauna Loa, in Hawaii archipelo. This data was also obtained from the website of the Earth System Research Laboratory (http://www.esrl.noaa.gov/gmd/ccgg/trends/). Longer term data is only available from indirect measurements, i.e. ice core data. The data for long time periods was also retrieved from the web site of the Carbon Dioxide Information Analysis Center (CDIAC) (http://cdiac.esd.ornl.gov/ftp/trends/co2/siple2.013, http://cdiac.ornl.gov/ftp/trends/co2/lawdome.combined.dat), the National Oceanic and Atmospheric Administration (NOAA) (ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_annmean_mlo.txt, ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/maud/edml-co2-2005.txt, ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_mm_mlo.txt) and from [2][7].

B Exact solution of the ODE system ((14),(15))

This appendix provides the exact derivation of the system of Eqs. (14) and (15), thus justifying the ansatz (16) and (17) used.

First, we combine Eqs. (14) and (15) into a single equation:

$$\frac{dA}{dt} L(t)^{-\eta-\gamma} A(t)^{-\theta} - \frac{dL}{dt} L(t)^{-1} A(t)^{-1+\alpha} = 0. \tag{B.1}$$

Without loss of generality, we can set $e = f = 1$ by defining appropriately the units of A and L. Separating the variables and integrating lead to

$$\frac{1}{2-\alpha-\theta} A(t)^{2-\alpha-\theta} - \frac{1}{\eta+\gamma} L(t)^{\eta+\gamma} = c'. \tag{B.2}$$

Looking for the large time asymptotic regime for which $L(T)$ and $A(t)$ (which are assumed to be monotonously increasing) become much larger that the constant c', we can solve for $A(t)$ and $L(t)$ by neglecting c':

$$L(t) = \left[\frac{1}{2-\alpha-\theta} A(t)^{2-\alpha-\theta} (\eta+\gamma)\right]^{1/(\eta+\gamma)} \tag{B.3}$$

$$= c_2 A(t)^{\frac{2-\alpha-\theta}{\eta+\gamma}}. \tag{B.4}$$

Plugging this into equation (14) leads to

$$\frac{dA}{dt} = c_2 A(t)^{2-\alpha}. \tag{B.5}$$

[7] Leaving out "Bern" measurements.

By separating variables and subsequently integrating, we get:

$$A(t)^{\alpha-2}dA = c_2 dt, \tag{B.6}$$

$$\frac{1}{\alpha-1}A(t)^{\alpha-1} = c_2 t + c_2', \tag{B.7}$$

$$A(t) = \left[(1-\alpha)c_2(\frac{-c_2'}{c_2} - t)\right]^{-1/(1-\alpha)} \tag{B.8}$$

$$\Leftrightarrow A(t) = A_0(t_c - t)^{-1/\mu}, \tag{B.9}$$

with $\mu = 1-\alpha > 0$ since $\alpha < 1$.

Similarly, we find the solution for $L(t)$:

$$A(t) = \left[\frac{1}{\eta+\gamma}L(t)^{\eta+\gamma}\right]^{1/(2-\alpha-\theta)} \tag{B.10}$$

$$= c_3 L(t)^{\frac{\eta+\gamma}{2-\alpha-\theta}}. \tag{B.11}$$

Plugging this into Eq. (15) leads to

$$\frac{dL}{dt} = c_3' L(t)^{\frac{(\eta+\gamma)(1-\alpha)}{2-\alpha-\theta}+1} \tag{B.12}$$

$$=: c_3' L(t)^{\kappa+1} \quad \text{where } \kappa := \frac{(\eta+\gamma)(1-\alpha)}{2-\alpha-\theta}. \tag{B.13}$$

As before, we separate variables and integrate

$$L(t)^{-\kappa-1}dL = c_3' dt, \tag{B.14}$$

$$\frac{1}{-\kappa}L(t)^{-\kappa} = c_3' t + c_3'', \tag{B.15}$$

$$L(t) = \left[\kappa c_3' \left(\frac{-c_3''}{c_3'} - t\right)\right]^{-1/\kappa} \tag{B.16}$$

$$\Leftrightarrow L(t) = L_0(t_c - t)^{-1/\kappa}, \tag{B.17}$$

with $\kappa = \frac{\eta+\gamma}{2-\alpha-\theta}(1-\alpha)$.

Of course, the solution for $L(t)$ could be directly obtained using (B.4) and (B.9), and reciprocally.

For a general mathematical rigorous theory of ordinary differential equations exhibiting finite-time singular behaviors, see [12].

C Calculation of the exponent φ

Let us give some intermediate steps towards the solution of Eq. (20).

$$\frac{Y(t)}{A(t)^{\xi}} =^{(10)} \frac{K(t)^{\alpha}(A(t)L(t))^{1-\alpha}}{A(t)^{\xi}} \tag{C.1}$$

$$=^{(12)} L(t)A(t)^{1-\alpha-\xi} \tag{C.2}$$

$$=^{(16,17)} L_0(t_c-t)^{-1/\kappa}\left[A_0(t_c-t)^{-1/\mu}\right]^{1-\alpha-\xi} \tag{C.3}$$

$$= L_0A_0(t_c-t)^{-1/\kappa-(1-\alpha-\xi)/\mu} \tag{C.4}$$

$$=^{!} C_0(t_c-t)^{-1/\varphi}. \tag{C.5}$$

Hence,

$$\varphi = \frac{1}{1/\kappa - \xi/\mu + 1}, \tag{C.6}$$

using $\mu = 1-\alpha$ given by (18).

References

1. A. Akaev, V. Sadovnichy, A. Korotayev, Eur. Phys. J. Special Topics **205**, 355 (2012)
2. J.M. Barnola, M. Anklin, J. Porcheron, D. Raynaud, J. Schwander, B. Stauffer, Tellus B **47**(1–2), 264 (1995)
3. R. Biggs, S.R. Carpenter, W.A. Brock, Proc. Natl. Acad. Sci. USA **106**(3), 826 (2009)
4. J.G. Canadell, C. Le Quere, M.R. Raupach, C.B. Field, E.T. Buitenhuis, P. Ciais, T.J. Conway, N.P. Gillett, R.A. Houghton, G. Marland, Proc. Natl. Acad. Sci. **104**(47) 18866 (2007)
5. M.R. Chertow, J. Industr. Ecol. **4**(4), 13 (2000)
6. C.W. Cobb, P.H. Douglas, Amer. Econ. Rev. **18**(1), 139 (1928)
7. V. Dakos, M. Scheffer, E.H. van Nes, V. Brovkin, V. Petoukhov, H. Held, Proc. Natl. Acad. Sci. USA **105**(38), 14308 (2008)
8. J.M. Drake, B.D. Griffen, Nature **456**(September), 456 (2010)
9. T. Garrett, Climatic Change **104**(3), 437 (2011)
10. S. Gluzman, D. Sornette, Phys. Rev. E **6601**(016134), U315 (2002)
11. N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Perseus Publishing, 1992)
12. A. Goriely, J. Differen. Eqns. **161**(2), 422 (2000)
13. C.A.S. Hall, J.W. Day, Jr., Amer. Scientist **97**, 230 (2009)
14. K. Ide, D. Sornette, Physica A **307**(1–2), 63 (2002)
15. A. Johansen, D. Sornette, Physica A: Stat. Mech. Appl. **294**(3-4), 465 (2001)
16. A. Korotayev, J. World-Syst. Res. **11**(1), 79 (2005)
17. A. Korotayev, A.S. Malkov, D. Khaltourina, *Introduction to Social Macrodynamics: Secular Cycles and Millennial Trends* (URSS, 2006)
18. M. Kremer, Q. J. Econ. **108**, 681 (1993)
19. D.H. Meadows, *The Limits to Growth; a Report for the Club of Rome's Project on the Predicament of Mankind* (Universe Books, 1972)
20. Organisation for Economic Co-operation and Development, *World Energy Outlook 2011*
21. R. Pielke, T. Wigley, C. Green, Nature **452**(7187), 531 (2008)
22. M.R. Raupach, J.G. Canadell, C. Le Quéré, Biogeosci. Discuss. **5**(4), 2867 (2008)

23. J. Rockstrom, W. Steffen, K. Noone, A. Persson, F.S. Chapin, E.F. Lambin, T.M. Lenton, M. Scheffer, C. Folke, H.J. Schellnhuber, B. Nykvist, C.A. de Wit, T. Hughes, S. van der Leeuw, H. Rodhe, S. Sorlin, P.K. Snyder, R. Costanza, U. Svedin, M. Falkenmark, L. Karlberg, R.W. Corell, V.J. Fabry, J. Hansen, B. Walker, D. Liverman, K. Richardson, P. Crutzen, J.A. Foley, Nature **461**(7263), 472 (2009)
24. D. Romer, *Advanced Macroeconomics*, 2nd ed. (McGraw-Hill/Irwin, 2000)
25. Royal Society and the US National Academy of Sciences, Climate Change: Evidence & Causes
26. S.G. Sammis, D. Sornette, Proc. Natl. Acad. Sci. USA **99**(Supp1), 2501 (2002)
27. M. Scheffer, J. Bascompte, W.A. Brock, V. Brovkin, S.R. Carpenter, V. Dakos, H. Held, E.H. van Nes, M. Rietkerk, G. Sugihara, Nature **461**(7260), 53 (2009)
28. D. Sornette, Proc. Natl. Acad. Sci. USA **99**(Supp1), 2522 (2002)
29. D. Sornette, *Why Stock Markets Crash (Critical Events in Complex Financial Systems)*, (Princeton University Press, 2003)
30. D. Sornette, *Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-organization and Disorder: Concepts and Tools (Springer Series in Synergetics)*, 2nd ed. (Springer, 2006)
31. P.A. Stephens, W.J. Sutherland, R.P. Freckleton, Oikos **87**, 185 (1999)
32. S.A. Umpleby, Population Env. **11**(3), 159 (1990)
33. P.-F. Verhulst, Mém. de l'Academie Royale des Sci. et Belles-Lettres de Bruxelles **18**, 1 (1845)
34. P.-F. Verhulst, Mém. de l'Academie Royale des Sci. et Belles-Lettres de Bruxelles **20**, 1 (1847)
35. H. von Foerster, P.M. Mora, L.W. Amiot, Science **132**(3436), 1291 (1960)
36. P.E. Waggoner, J.H. Ausubel, Proc. Natl. Acad. Sci. USA **99**(12), 7860 (2002)
37. S.R. Weart, *The Discovery of Global Warming: Revised and Expanded Edition (New Histories of Science, Technology, and Medicine)* (Harvard University Press, revised and expanded edition, 2008)
38. V.I. Yukalov, E.P. Yukalova, D. Sornette, Physica D **238**, 1752 (2009)
39. V.I. Yukalov, E.P. Yukalova, D. Sornette, Eur. Phys. J. Special Topics **205**, 313 (2012)
40. V.I. Yukalov, E.P. Yukalova, D. Sornette, Physica D **241**, 1270 (2012)
41. V.I. Yukalov, E.P. Yukalova, D. Sornette, Int. J. Bifurc. Chaos **24**(2), 1450021 (2014)